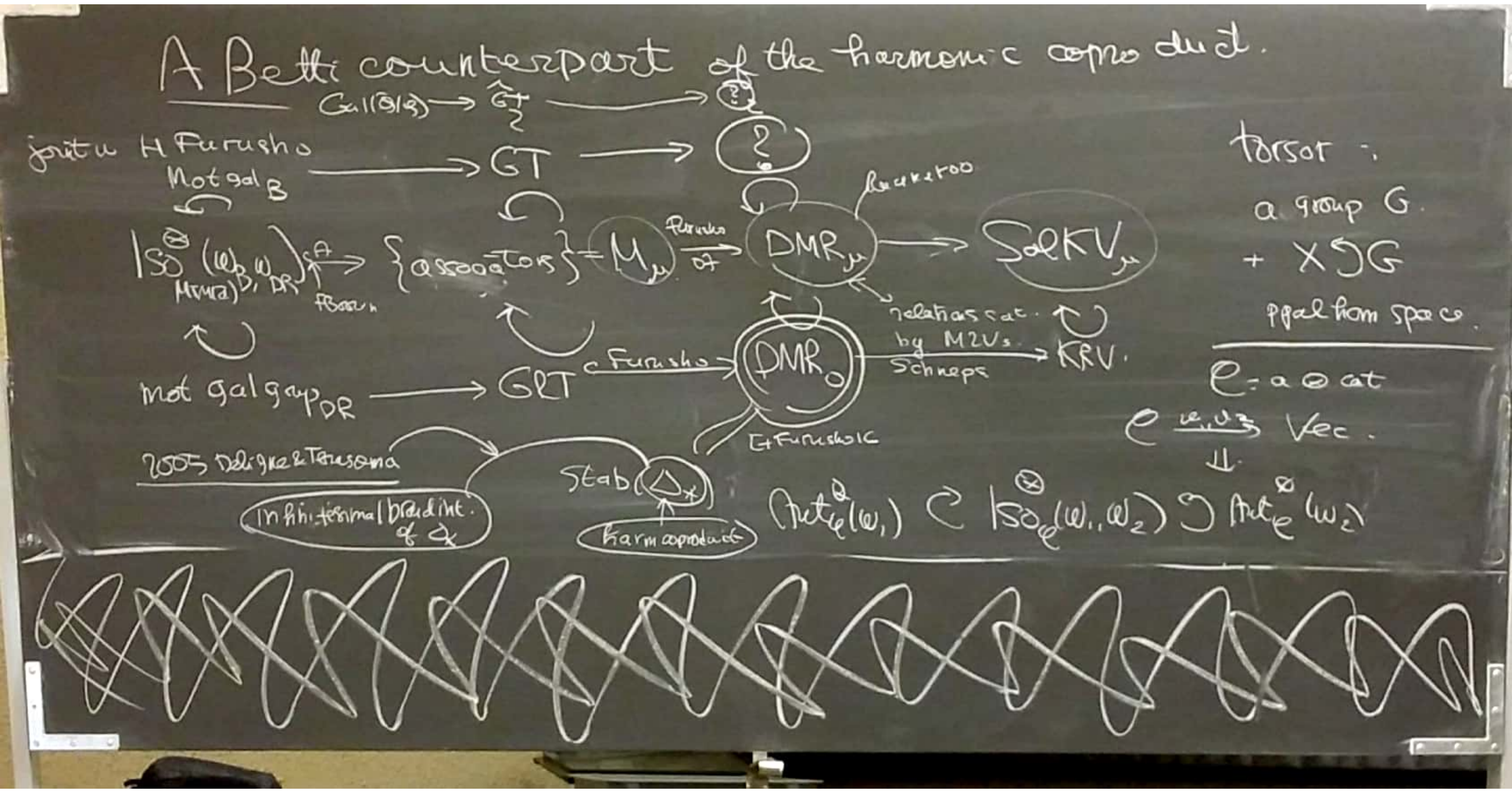
A Betti counterpart of the harmonic coproduct.
Gal(Q̄/Q) → GT̂
joint w H Furusho
mot gal B
GT
?
DMR_μ
SolKV_μ
{associators} = M_μ
Furusho 07
relations sat. by MZVs
Schneps
KRV
mot gal grp_DR
GRT
Furusho
DMR_0
2005 Deligne & Terasoma
Stab(Δ_*)
harm coproduct
torsor:
a group G
+ X ⟲ G
ppal hom space
C : a ⊗ cat
Vec

$G \subset X \circlearrowleft G'$ ppal hom.

$G' = \mathrm{Aut}_G(X)$.

① Recall stab interp. of DMR_0 & Δ_*.

② Construct new object $\Delta_*(\mu)$ + compute $\Delta_*(\mu)$.

③ how to construct Betti versions of DMR_0

④ coal bracket of Δ_*

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① x_0, x_1 = NC variables

$$\mathbb{Q}\langle x_0, x_1\rangle \longleftrightarrow \mathbb{Q} \oplus \mathbb{Q}\langle x_0, x_1\rangle x_1 = \mathcal{W}_e^{DR}$$

$$\cup$$

$$\boxed{\begin{matrix}\mathbb{Q}\langle x_0, x_1\rangle^\times \\ \| \\ \mathrm{pdt}\ \circledast\end{matrix}} \circlearrowright_{\circledast} \mathbb{Q}\langle x_0, x_1\rangle \longrightarrow \mathcal{W}_e^{DR}$$

mode out $\mathbb{Q}\langle x_0, x_1\rangle x_0$

$$G.H = GH(x_0, G^{-1}x_1G).$$

$$\cup \qquad \circledcirc$$

$$(\exp(\mathrm{Lie}_2), \circledast)$$

$$\mathrm{Lie}_2 = \mathrm{Lie}(x_0, x_1)$$

$$\Theta(g) = \underbrace{\Gamma(g)}_{\exp\left(-\sum \frac{(-1)^{n-1}}{n}(g|x_0^{n-1}x_1)x_1^n\right)} g \exp(-(g|x_0)x_0) \Longrightarrow \text{action } S^Y_{\Theta(g)} \text{ on } (\exp \mathrm{Lie}_2, \circledast) \text{ on vspace } \mathcal{W}_e^{\vee DR}$$

$$\boxed{(\exp \mathrm{Lie}_2, \circledast)} \xrightarrow{\Theta} (\mathbb{Q}\langle\langle x_0, x_1\rangle\rangle^{\times}, \circledast) \circlearrowright \boxed{\mathcal{W}_e^{DR}}$$

$$g \longmapsto S^Y_g$$

$$h \longmapsto S^Y_{\Theta(g)}$$

$$\Downarrow$$

act of $(\exp \mathrm{Lie}_2, \circledast)$ on $\mathrm{Hom}_{\mathbb{Q}}(\mathcal{W}_e^{DR}, (\mathcal{W}_e^{DR})^{\otimes 2}) \ni \Delta_*$

<u>Thm</u> (E-Furusho, 2016) $\mathrm{DMR}_0(\circledast\, \mathbb{C}^2) = \left\{ g \in (\exp \mathrm{Lie}_2, \circledast) \;\middle|\; \Delta_* S^Y_{\Theta(g)} = (S^Y_{\Theta(g)})^{\otimes 2} \circ \Delta_* \right\}$

where $\Delta_*^{DR} \in Hom_{\mathbb{Q}}(\mathcal{W}_\ell^{DR}, \mathcal{W}_\ell^{DR \hat{\otimes} 2})$

is the algebra morphism s.t.

$$\Delta_*^{DR}(y_n) = y_n \otimes 1 + 1 \otimes y_n + \sum_{k=1}^{n-1} y_k \otimes y_{n-k} \quad (n \geq 1)$$

and $y_n = x_0^{n-1} x_1$

DMR_μ

Lemma:

$$\begin{array}{ccc} G & \circlearrowright & X = \text{set.} \\ \circlearrowleft & & \cup \\ S_{x_0} & & x_0 \\ \| & & \\ \text{stab of } x_0. & & \end{array}$$

$(\exp \mathfrak{lie}_2, \circledast)$

$Hom(\mathcal{W}_\ell^{DR}, \mathcal{W}_\ell^{DR \hat{\otimes} 2})$

Δ_*^{DR}

$\{\Delta_1^B(\mu)\}$

S_{x_0} left action on G.

Assume that $H \subset G$ is a torsor for this left action.

Then: $H^{-1} \cdot x_0 = \{1 \text{ point}\} = \{x_1\}$

and $H = \{g \in G \mid g \circ x_1 = x_0\}$

$G \circlearrowright X \circlearrowleft G'$ pro-alg. group, free

$G' = \mathrm{Aut}_G(X)$.

1. Recall stab interp. of DMR_0 & Δ_*
2. Construct new object $\Delta_*(\mu)$ + compute $\Delta_*(\mu)$
3. how to construct Betti versions of DMR_0
4. coalgebra grad. of Δ_*

<u>Lemma:</u> $\Delta_*^{B}(\mu) :=$ common value of all

$$(S^{Y}_{\theta(g)})^{-1 \otimes 2} \circ \Delta_* \circ S^{Y}_{\theta(g)}$$

$g \in DMR_{\mu}$

Then

(1) $\Delta_*^{B}(\mu)$ is an algebra morphism $\hat{\mathcal{W}}_e^{DR} \to \hat{\mathcal{W}}_e^{DR \otimes 2}$.

(2) it is scaling equivariant

$$\begin{array}{ccc} \mathbb{Q}\langle x_0, x_1\rangle & \xrightarrow{h_\cdot} & \mathbb{Q}\langle x_0, x_1\rangle \\ x_0, x_1 & & h x_0, h x_1 \\ \cup & & \\ \mathcal{W}_e^{DR} & \xrightarrow{h_\cdot} & \mathcal{W}_e^{DR} \end{array}$$

$$\begin{array}{ccc} \mathcal{W}_e^{DR} & \xrightarrow{\Delta^{DR}_*(\mu)} & \mathcal{W}_e^{DR \otimes 2} \\ \downarrow h & & \downarrow h^{\otimes 2} \\ \mathcal{W} & \xrightarrow{\Delta_*(h\mu)} & \mathcal{W} \end{array}$$

$\Delta_*^{B}(\mu) = \Delta_*^{DR} + \mu\, \Delta_*(1) + \mu^2\, \Delta_*(2) + \ldots$

Betti analogues of $\mathcal{W}_\ell^{DR} \subset \widehat{\mathcal{W}}_\ell^{DR}$

let F_2 = free group, X_0, X_1

$\boxed{\begin{array}{l} \text{presentation:} \\ \text{generators are } X_0^k(X_1-1),\ k\in\mathbb{Z} \\ \qquad\qquad\qquad X_1^{-1}-1 \\ 1 \text{ relation reflecting } X_1.X_1^{-1}=1 \end{array}}$

$$\begin{tikzcd}
 & \mathbb{Q}F_2 \arrow[d, "\wr"] & \mathbb{Q}\oplus\mathbb{Q}F_2(X_1-1) \arrow[l, hook'] \arrow[r, equal] & \boxed{\mathcal{W}_\ell^{B,\,disc}} \arrow[d, "\wr"] \\
X_0, X_1 \in \arrow[d] & (\mathbb{Q}F_2)^{unip} = \varprojlim(\mathbb{Q}F_2/I^n) \arrow[d, "\mathrm{iso}_\mu^\ell", "\wr"'] & & \mathcal{W}_\ell^{B,\,unip} = \mathbb{Q}\oplus(\mathbb{Q}F_2)^{unip}(X_1-1) \arrow[ll, hook'] \arrow[d, "\wr", "\mathrm{iso}_\mu^\ell"'] \\
e^{\mu x_0}, e^{\mu x_1} \in & \widehat{U}(Lie_2) = \mathbb{Q}\langle\langle x_0, x_1\rangle\rangle & & \widehat{\mathcal{W}}_\ell^{DR} \arrow[ll, hook']
\end{tikzcd}$$

where $\Delta_*^{DR} \in \mathrm{Hom}_{\mathbb{Q}}(\mathcal{W}_\ell^{DR}, \mathcal{W}_\ell^{DR \otimes 2})$

is the algebra morphism s.t.

$$\Delta_*^{DR}(y_n) = y_n \otimes 1 + 1 \otimes y_n + \sum_{k=1}^{n-1} y_k \otimes y_{n-k} \quad (n \geq 1)$$

and $y_n = x_0^{n-1} x_1$. $(-e_0^{n-1} e)$

$$gr(\mathcal{W}_\ell^{B,disc}) = \mathcal{W}_\ell^{DR}$$

Lemma: $\exists!$ alg morphism

$$\mathcal{W}_\ell^{B,disc} \xrightarrow{\Delta^{B,disc}} (\mathcal{W}_\ell^{B,disc})^{\otimes 2}$$

$$\xi_k = X_0^k (X_1 - 1) \longmapsto \xi_k \otimes 1 + 1 \otimes \xi_k - \sum_{i=1}^{k-1} \xi_i \otimes \xi_{k-i}, \quad k \in \mathbb{Z}.$$

$$\xi^- = X_1^{-1} - 1 \mapsto \xi^- \otimes 1 + 1 \otimes \xi^- + \xi^- \otimes \xi^-$$

It's δ cocomm coass. | extends to a morphism $\mathcal{W}_\ell^{B,unip} \to (\mathcal{W}_\ell^{B,unip})^{\otimes 2}$

compat w filtration

$(X_0 - 1)^k (X_1 - 1) \to$

$$gr(\Delta^{B,disc}) = \Delta^{DR}.$$

$G \subset X \circlearrowleft G'$ pro-hom.

$G' = \mathrm{Aut}_G(X)$.

1) Recall stab interp of DMR_0 & Δ_*

2) Construct new object $\Delta_*(\mu)$
+ compute $\Delta_*(\mu)$

3) how to construct Betti versions of DMR_0

4) [illegible] of Δ_*

Thm: ∃ comm diag

$$\begin{tikzcd} W_e^{DR} \arrow[r, "\Delta_*^{B}(\mu)"] \arrow[d, "\mathrm{Ad}\left(\frac{\log x_1}{x_1 - 1}\right) \circ \mathrm{iso}_\mu"'] & W_e^{DR \otimes 2} \arrow[d, "\left(\mathrm{Ad}\left(\frac{\log x_1}{x_1 - 1}\right) \circ \mathrm{iso}_\mu\right)^{\otimes 2}"] \\ W_e^{B,\mathrm{unip}} \arrow[r, "\Delta_*^{B,\mathrm{disc}}"'] & W_e^{B,\mathrm{unip} \otimes 2} \end{tikzcd}$$

$$\begin{array}{c} \widehat{DM}_0 \\ \uparrow \\ DM_0 \\ \downarrow \end{array}$$

$DM_0^{(k)} = \mathrm{Aut}(\Delta_*^{B}(\mu))^{(k)} \cong \mathrm{Aut}(\Delta_*^{B,\mathrm{disc}})(k)$

$M_\mu \hookrightarrow DMR_\mu$

$\Delta_*^{B}(\mu) = \bar{\Phi}^* \Delta_*^{DR}$

DT interp of Δ_{τ}^{PR}

∞ parebraids on 4pt on S^2 — $\mathbb{P}_4 = \mathfrak{f}_2$

5pt. — $\mathbb{P}_5 = \mathfrak{t}_4 / Z$

$w_\Gamma = \mathbb{Q} \oplus x_1 \mathbb{Q}\langle x_0, x_1 \rangle$

co-parenthesized braids 4pt on $\mathbb{R}^2$

$$\mathbb{P}_5 \underset{e}{\overset{pr_i}{\rightleftarrows}} \mathbb{P}_4 = \mathfrak{f}_2 = \langle e_0, e_1 \rangle \qquad i = 1..5$$

$(e_{ij} \ _{i,j \in [1,5]})$

$$\begin{array}{ccccc} C_{\Gamma}(\mathbb{P}_4\text{-mod})^{\otimes 2} & \xrightarrow{\ominus} & \left\{ \mathbb{P}_5 \circlearrowright V \underset{\beta}{\overset{\alpha}{\rightleftarrows}} W \ \middle| \ \beta\alpha = e(e_1) \right\} & \xrightarrow{\ominus} & w_e\text{-mod} \\ \downarrow \mathrm{can} & & & & \uparrow \Delta_{\tau}^{PR, err.} \\ (\mathbb{Q}\langle x_0, x_1 \rangle\text{-mod})^{\otimes 2} & & \xrightarrow{\mathrm{res}} & & (w_\Gamma)^{\otimes 2}\text{-mod} \end{array}$$

$\boxed{W, \ x e_1 \circ w = \alpha\, e(x)\, \beta(w)}$

$G \subset X \circlearrowleft G'$ ppel-hom.

$G' = \mathrm{Aut}_G(X)$.

1. Recall stab interp of DMR_0 & Δ_*
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4. ooal Mardiut of Δ_*

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Ac

$$\mathbb{P}_3 \curvearrowright \mathrm{Hom}_{\mathbb{P}_5}\left(J(pr_5), (pr_1 \otimes pr_2)^*(\mathbb{C})\right) \rightleftarrows \mathbb{C}$$

$$J(pr_5) = \mathrm{Ker}\left(U(pr_5) : U(\mathbb{P}_5) \to U(\mathbb{P}_4)\right)$$

$$\begin{array}{ccc} \omega_r^{\mathrm{OR}} & \xrightarrow{\Delta^{B,r}_{*,1}(\mu)} & \omega_r^{\mathrm{OR} \otimes 2} \\ \mathrm{iso} \downarrow & & \downarrow \mathrm{iso} \\ \omega_r^{B} & \xrightarrow[\Delta_r^{B,\mu}]{} & \omega_r^{B \otimes 2} \end{array}$$